Examples
Linear codes

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1. Let $q = 2$. Let $S = \{0001, 0010, 0100\}$ be a subset of a vector space V over $\mathbf{F}_2$. Find the linear span of S, $\langle S \rangle$.

Solution. Write the span as

$$\lambda_1(0001) + \lambda_2(0010) + \lambda_3(0100)$$

where $\lambda_i \in \mathbf{F}_2$, $i = 1, 2, 3$.

$\lambda_1\lambda_2\lambda_3$		vector
000	$\rightarrow$	0000
100	$\rightarrow$	0001
010	$\rightarrow$	0010
001	$\rightarrow$	0100
011	$\rightarrow$	$0010 + 0100 = 0110$
101	$\rightarrow$	$0001 + 0100 = 0101$
110	$\rightarrow$	$0001 + 0010 = 0011$
111	$\rightarrow$	$0001 + 0020 + 0100 = 0111$

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2. Let $q = 3$ and $S = \{12101, 20110, 01122, 11010\}$. Find a basis for $C = \langle S \rangle$.

Solution. We have

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Reduce by elementary row operations our A into the row echelon form. Thus,

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then $\{12101, 01122, 00001\}$ is a basis for C.

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3. Let $q = 2$ and $S = \{11101, 10110, 01011, 11010\}$ Find a basis for $C = \langle S \rangle$.

Solution. Form a matrix A according to Algorithm 4.2, and reduce it into row echelon form.

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the leading columns, which are those that contain the first one in each row, are 1, 2 and 4. Therefore $\{11101, 10110, 11010\}$ forms a basis for C.

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4. Let C be the binary $[5,3]$-linear code with the generator matrix

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Encode the message $\mathbf{u} = 101$ and find the information rate of C.

Solution. The message $\mathbf{u}$ is encoded into $\mathbf{v}$ as,

$$\mathbf{v} = \mathbf{u}G = (101)\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} = (1 \quad 0 \quad 0 \quad 1 \quad 1)$$

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The information rate of C is $\frac{3}{5}$

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5. Find the cosets of the binary linear code

$$C = \{0000, 1011, 0101, 1110\}$$

Solution. We find cosets one after another, and see whether they are a new one.

$0000 + C \quad \to \quad 0000,\ 1011,\ 0101,\ 1110 \quad \to \quad \text{I}$
$0001 + C \quad \to \quad 0001,\ 1010,\ 0100,\ 1111 \quad \to \quad \text{II}$
$0010 + C \quad \to \quad 0010,\ 1001,\ 0111,\ 1100 \quad \to \quad \text{III}$
$0011 + C \quad \to \quad \text{IV}$
$0100 + C \quad \to \quad \text{II}$
$0101 + C \quad \to \quad \text{I}$
$0110 + C \quad \to \quad \text{IV}$
$0111 + C \quad \to \quad \text{III}$
$1000 + C \quad \to \quad 1000,\ 0011,\ 1101,\ 0110 \quad \to \quad \text{IV}$
$1001 + C \quad \to \quad \text{III}$
$1010 + C \quad \to \quad \text{II}$
$1011 + C \quad \to \quad \text{I}$
$1100 + C \quad \to \quad \text{III}$
$1101 + C \quad \to \quad \text{IV}$
$1110 + C \quad \to \quad \text{I}$
$1111 + C \quad \to \quad \text{II}$

Thus there are four cosets, and they are the I, II, III and IV as shown.

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